

The comparison Gelfand-Tsetlin-Molev and Gelfand-Tsetlin-Zhelobenko bases for \mathfrak{sp}_{2n}

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A construction of Gelfand-Tsetlin type base vectors in a finite-dimensional representation of \mathfrak{sp}_{2n} was firstly obtained in 60-th by Zhelobenko [3]. But the final construction was obtained only in the year 1998 by Molev [5], who gave a construction of Gelfand-Tsetlin type base vectors and derived formulas for the action of generators of the algebra in this base. These two approaches use different ideas.

In the present paper we compare these two approaches. Also we show that the Molev's base vectors can be obtained using a construction based on a relation between restriction problems $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ and $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$, analogous to the construction giving the Zhelobenko's base.

1 Introduction

In the year 1950 there appeared two short articles of Gelfand and Tsetlin [1], [2], where they gave explicit construction of finite-dimensional representation for the Lie algebras \mathfrak{gl}_n and \mathfrak{o}_n . More precise they gave indexation of base vectors and obtained formulas for the action of generators in the base. In these notes there were not given a derivation of the presented formulas. In the case of \mathfrak{gl}_n the derivation was published by Zhelobenko in the paper [3] (see also [4]).

A natural question is the construction of an analogue of the Gelfand-Tsetlin base for symplectic algebras. However the attempt to apply the ideas of Gelfand-Tsetlin in this situation meets a new phenomenon. The construction a base for the series of algebras g_n (where $g_n = \mathfrak{gl}_n$ or \mathfrak{sp}_{2n}) is based on investigations of the branching of an irreducible representation under the restriction of the algebra $g_n \downarrow g_{n-1}$. In the case \mathfrak{gl}_n the spectrum of possible g_{n-1} -irreps is simple and in the case of \mathfrak{sp}_{2n} can have multiplicities.

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To construct a Gelfand-Tsetlin type base for the symplectic algebra one has to find a way of construction of a base in the space of \mathfrak{sp}_{2n-2} -highest vectors in a given irrep of \mathfrak{sp}_{2n} .

For the first time the solution of this problem was given by Zhelobenko in [3] (see also [4]). However he managed not to find the formulas for the action of generators in his base.

The full solution of the problem of construction of the Gelfand-Tsetlin type base for the algebra \mathfrak{sp}_{2n} (including the derivation of the formula for the action of generators of the algebra in this base) was given by Molev in [5] (see also [6]). He used a new technique: the main step in the construction of the base was a construction of the action of the yangian $Y(\mathfrak{gl}_2)$ on the space of \mathfrak{sp}_{2n-2} -highest vectors with a fixed \mathfrak{sp}_{2n-2} -weight.

Mention that the indexation of the base vectors in the approaches of Molev and Zhelobenko is the same.

Thus there naturally appears a question: do these constructions define the same base or not. The answer to this question is negative. The reason is simple: these two constructions use different subalgebras $\mathfrak{sp}_{2n-2} \subset \mathfrak{sp}_{2n}$. More precisely, let \mathfrak{sp}_{2n} be a Lie algebra that preserves the symplectic form $\sum_{i=1, \dots, n} dx_{-i} \wedge dx_i$ in the space \mathbb{C}^{2n} with coordinates x_{-n}, \dots, x_n . Then Zhelobenko takes the subalgebra \mathfrak{sp}_{2n-2} that preserves the coordinates $x_{-n}, \dots, x_{-2}, x_2, \dots, x_n$ ¹, and Molev takes a subalgebra that preserves the coordinates x_{-n+1}, \dots, x_{n-1} . Thus the Gelfand-Tsetlin type bases are different.

But one can ask then the following question. What are the properties of the automorphism of an \mathfrak{sp}_{2n} -irrep that sends a vector of the base of Zhelobenko to the vector of the base of Molev corresponding to the same diagram. The answer is given in the Theorem 1. It turns out that this mapping preserves modulo multiplication by a constant the action of the so called raising and lowering operators that form the Mickelsson-Zhelobenko algebra (see its definition in Section 3.1).

As a byproduct we obtain isomorphisms between different Mickelsson-Zhelobenko algebras (see Lemma 4). Also we obtain an important result saying that the Molev's base, that was originally obtained using a construction of an action of a yangian on the multiplicity space, can be obtained as the Zhelobenko's base using a relation between the restriction problems $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ and $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ (see Corollary 2).

Since both approaches to the construction of the base use specific normalization it is reasonable to consider the vectors modulo multiplication by a constant

¹When we say that a subalgebra preserves some coordinates we mean that the subspace spanned by these coordinates is invariant and onto complement coordinates the algebra act as zero

2 The Zhelobenko's construction

2.1 The method of Z -invariants of Zhelobenko

In the present paper we use a realization of \mathfrak{sp}_{2n} in which it is spanned by matrices

$$F_{i,j} = E_{i,j} - \text{sign}(i)\text{sign}(j)E_{-j,-i}, \quad i, j = -n, \dots, n,$$

The only linear relation between these matrices is the following $F_{i,j} = -\text{sign}(i)\text{sign}(j)F_{-j,-i}$, also

$$F_{i,j} \text{ is } \begin{cases} i = j \text{ a Cartan element ,} \\ i > j \text{ a negative root element ,} \\ i < j \text{ a positive root element} \end{cases}$$

On a dense open subset of Sp_{2n}^0 in the group Sp_{2n} we have a Gauss decomposition

$$\begin{aligned} Sp_{2n}^0 &= Z^- D Z, \quad X = \zeta \delta z, \\ X &\in Sp_{2n}, \quad \zeta \in Z^-, \quad \delta \in D, \quad z \in Z, \end{aligned} \tag{1}$$

where Z^- is a subgroup of lower-triangular matrices from Sp_{2n} with unit on the diagonal, D is a subgroup of diagonal matrices from Sp_{2n} , Z is a subgroup of upper-triangular matrices from Sp_{2n} with units on the diagonal. Let $z = (z_{i,j})$, $\delta = \text{diag}(\delta_i)$. There exists an action of Sp_{2n} on the function on Z by the following ruler. Let us be given a function on Z of type $f(z) = f(z_{i,j})$, $i < j$. For $X \in Sp_{2n}$ put

$$\begin{aligned} (Xf)(z) &= \alpha(\tilde{\delta})f(\tilde{z}), \quad zX = \tilde{\zeta}\tilde{\delta}\tilde{z} \\ \alpha(\delta) &= \delta_{-1}^{r_{-1}} \dots \delta_{-n}^{r_{-n}}, \end{aligned}$$

and r_i are some non-negative integers. To obtain below a realization of an irrep with the highest weight $[m_{-n}, \dots, m_{-1}]$ we must put $r_{-i} = m_{-i} - m_{-i+1}$, $r_{-1} = m_{-1}$. Here m_{-i} is an eigenvalue on the highest vector of the operator $F_{-i,-i}$.

All function on Z form a reducible representation, to obtain an irrep with the highest weight $[m_{-n}, \dots, m_{-1}]$ let us do the following. Let $L_{i,j}$, $i < j$ be an operator on the functions $f(z)$ which makes a left infinitesimal shift of the function $f(z)$ to $F_{i,j}$. Then an irrep with the highest weight $[m_1, \dots, m_n]$ is formed by functions that satisfy the *indicator system*

$$L_{-n,-n+1}^{r_{-n}+1}f = 0, \dots, L_{-1,1}^{r_{-1}+1}f = 0.$$

In an explicit form it is a system of PDE.

The construction of the Gelfand-Tsetlin type base is based on the investigation of the branching of an irrep under the restriction of the algebra. The Zhelobenko's realization is very useful in the investigation of this problem.

Let V be an \mathfrak{sp}_{2n} -irrep with the highest weight $[m_{-n}, \dots, m_{-1}]$. We restrict the algebra $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$, where \mathfrak{sp}_{2n-2} preserves coordinates $\pm n, \dots, \pm 2$. Then V decomposes into the direct sum of \mathfrak{sp}_{2n-2} -irreps V_μ with the highest weight μ , every such irrep occurs with the multiplicity m_μ :

$$V = \bigoplus_{\mu} m_\mu V_\mu \quad (2)$$

It is clear that v is a \mathfrak{sp}_{2n-2} -highest vector if and only if $Z_{Sp_{2n-2}}v = v$, where $Z_{Sp_{2n-2}}$ is a subgroup in Z formed by elements from Sp_{2n-2} . In [3], [4] that the function f is a $Z_{Sp_{2n-2}}$ -invariant if and only if

$$f = f(z_{-n,-1}, \dots, z_{-2,-1}, z_{-n,1}, \dots, z_{-1,1}).$$

2.2 The relation between the problems of restriction $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ and $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$

Consider the algebra \mathfrak{gl}_{n+1} that acts in the space with coordinates $-n, \dots, -1, 1$. For this algebra also there exists a realization of representations on the functions on the subgroup in GL_{n+1} of upper-triangular matrices with units in the diagonal. An irrep is described by an indicator system (see [3]).

Consider the subalgebra \mathfrak{gl}_{n-1} , that preserve the coordinates $-n, \dots, -2$. Let us be given an irrep of \mathfrak{gl}_{n+1} with the highest weight $[m_{-n}, \dots, m_{-1}, 0]$ and consider the description of the branching of this representation under the restriction $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$.

It turns out that (see [3]) \mathfrak{gl}_{n-1} -highest weight vectors are functions

$$f = f(z_{-n,-1}, \dots, z_{-2,-1}, z_{-n,1}, \dots, z_{-1,1}).$$

To obtain an irrep with the highest weight written above one must consider the function that satisfies the indicator system. It turns out that being restricted to the functions $f = f(z_{-n,-1}, \dots, z_{-2,-1}, z_{-n,1}, \dots, z_{-1,1})$ the indicator system for \mathfrak{gl}_{n+1} with the highest weight $[m_{-n}, \dots, m_{-1}, 0]$ and for \mathfrak{sp}_{2n} with the highest weight $[m_{-n}, \dots, m_{-1}]$ coincide. Thus the sets of polynomials in variables $z_{-n,-1}, \dots, z_{-2,-1}, z_{-n,1}, \dots, z_{-1,1}$ that define the \mathfrak{gl}_{n-1} -highest and the \mathfrak{sp}_{2n} -highest vectors coincide. Thus the restriction problems $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ and $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ are equivalent.

2.3 The Gelfand-Tsetlin-Zhelobenko base

There exist a base in the space of \mathfrak{gl}_{n-1} -highest vectors, called the Gelfand-Tsetlin base, whose elements are index by tableaux (see [4])

$$\begin{array}{ccccccc} m_{-n,n} & m_{-n+1,n} & \dots & m_{-1,n} & 0 \\ m'_{-n,n} & m'_{-n+1,n} & \dots & m'_{-1,n} & \\ m_{-n,n-1} & \dots & m_{-2,n-1} & & \end{array} \quad (3)$$

where $m_{-i,n} = m_{-i}$. Denote the tableau (3) as (m) .

We do not discuss a normalization since in the present paper we consider vectors corresponding to tableaux modulo multiplication on a constant.

Take polynomials corresponding to these vectors. According to the previous Section they define all possible \mathfrak{sp}_{2n-2} -highest vectors. Thus we obtain a base in the space of \mathfrak{sp}_{2n-2} -highest vectors indexed by tableau (3).

Thus in an \mathfrak{sp}_{2n} -irrep we obtain a Gelfand-Tsetlin-Zhelobenko base, whose elements are index by tableaux of type

$$\begin{array}{ccccccc} m_{-n,n} & m_{-n+1,n} & \dots & m_{-1,n} & 0 \\ m'_{-n,n} & m'_{-n+1,n} & \dots & m'_{-1,n} & \\ m_{-n,n-1} & \dots & m_{-2,n-1} & & 0 \\ m'_{-n,n-1} & \dots & m'_{-2,n-1} & & \\ m_{-n,n-2} & m_{-3,n-2} & 0 & & \\ \dots & & & & \\ & m_{-n,1} & 0 & & \\ & m'_{-n,1} & & & \end{array} \quad (4)$$

Denote this tableau as $[m]$.

Denote the weight component with the number $-k$ as $weight_{-k}[m]$.

Lemma 1.

$$weight_{-(n-k+1)}[m] = 2 \sum m'_{i,k} - \sum m_{i,k-1} - \sum m_{i,k} \quad (5)$$

2.4 A realization on the functions on the whole group

Below we need another realization of a representation. This is a realization in the space of the function on the whole group Sp_{2n} . Onto the function $f(g)$ the element $X \in Sp_{2n}$ acts as follows

$$(Xf)(g) = f(gX).$$

The functions that form an irrep with a given highest weight are selected by some conditions (see [3]).

Let x_i^j and $\frac{\partial}{\partial x_i^j}$, $i, j = -n, \dots, n$ be a function of a matrix element on the space GL_{2n} of all invertible $(2n \times 2n)$ -matrices² and a differential operator on the space of such functions. Below we restrict the functions onto $Sp_{2n} \subset GL_{2n}$. Thus there appear relations between x_i^j , but we do not impose relations between differential operators.

Let us define the operators

$$a_i^j = x_i^j, \quad a_i^{+,j} = \frac{\partial}{\partial x_i^j}, \quad i, j = -n, \dots, n$$

Then

$$F_{i,j} = \sum_k \left(a_i^k a_j^{+,k} - \text{sing}(i) \text{sign}(j) a_{-j}^k a_{-i}^{+,k} \right).$$

Also define

$$a_{i_1, \dots, i_k} := \det(a_i^j)_{i=i_1, \dots, i_k}^{j=-n, \dots, -n+k-1}$$

One can easily check that the vector

$$\prod_{k=-n}^{-1} (a_{-n, \dots, -k})^{m_{-k} - m_{-k+1}} 1, \quad (6)$$

where 1 is a constant function, is the highest vector.

One can realize representation \mathfrak{gl}_{n+1} in the space of the functions on GL_{n+1} and also define a_i^j and $a_i^{+,j}$, but for $i, j = -n, \dots, -1, 1$. The generators $E_{i,j}$ act by formulas

$$E_{i,j} = \sum_k a_i^k a_j^{+,k}.$$

The formula (6) defines also a \mathfrak{gl}_{n+1} -highest vector of weight $[m_{-n}, \dots, m_{-1}, 0]$.

3 The Mikelsson-Zhelobenko algebra

3.1 The definition

The definition of the Mikelsson-Zhelobenko algebra is given in [7] (see also [6],[8]).

Let us be given a reductive Lie algebra \mathfrak{g} and its reductive subalgebra \mathfrak{k} . Consider the triangular decomposition

$$\mathfrak{k} = \mathfrak{k}_- \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_+,$$

²Thus there are no relations between these functions

Denote the set of positive roots for the algebra \mathfrak{k} as Δ_+ , the element of \mathfrak{k}_+ corresponding to the root $\alpha \in \Delta_+$ denote as e_α . There exist also elements $e_{-\alpha} \in \mathfrak{k}_-$.

Let R be a field of fractions of $U(\mathfrak{k}_0)$, put $U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}_0)} R$. Let $J' = U'(\mathfrak{g})\mathfrak{k}_+$.

Define

$$M(\mathfrak{g}, \mathfrak{k}) := U'(\mathfrak{g})/J'.$$

The the Mikelsson-Zhelobenko algebra is defined as follows

$$Z(\mathfrak{g}, \mathfrak{k}) := \{v \in M(\mathfrak{g}, \mathfrak{k}) : \mathfrak{k}^+ v = 0\}.$$

One can describe the generator of this algebra. For $\alpha \in \Delta_+$ put

$$p_\alpha = 1 + \sum_{k=1}^{\infty} e_{-\alpha}^k e_\alpha \frac{(-1)^k}{k!(h_\alpha + \rho(h_\alpha) + 1) \dots (h_\alpha + \rho(h_\alpha) + k)},$$

where $h_\alpha = [e_\alpha, e_{-\alpha}]$, ρ is a half-sum of positive roots. Choose a linear order $\alpha_1 < \dots < \alpha_m$ of roots from Δ_+ such that if one root is a sum of two others than it stands between them.

Put

$$p_{\mathfrak{g}, \mathfrak{k}} := p_{\alpha_1} \dots p_{\alpha_m}.$$

Proposition 1. *The operation of multiplication onto $p_{\mathfrak{g}, \mathfrak{k}}$ on the left side projects $M(\mathfrak{g}, \mathfrak{k})$ to $Z(\mathfrak{g}, \mathfrak{k})$.*

Proposition 2. *Let as a linear space $\mathfrak{g} = \mathfrak{k} \oplus \langle E_1, \dots, E_k \rangle$, then $Z(\mathfrak{g}, \mathfrak{k})$ is generated by elements $e_i := p_{\mathfrak{g}, \mathfrak{k}} E_i$*

Let us be given a representation V of the Lie algebra \mathfrak{g} . Consider the subspace V^+ that consists of \mathfrak{k} -highest vectors.

Proposition 3. *For $v \in V$ one has*

$$p_{\mathfrak{g}, \mathfrak{k}} v = \begin{cases} v, & \text{if } v \text{ is a } \mathfrak{k}\text{-highest vector,} \\ 0, & \text{if } v = e_- w, \text{ where } e_- \in \mathfrak{k} \text{ negative root element} \end{cases}$$

Proposition 4. *The action of \mathfrak{g} on V induces an action $Z(\mathfrak{g}, \mathfrak{k})$ on V^+ .*

3.2 The algebras used in the paper

3.2.1 $Z^{Zh}(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$

Let the algebra \mathfrak{gl}_{n+1} act in the space with coordinates $-n, \dots, -2, -1, 1$, and let \mathfrak{gl}_{n-1} be a subalgebra that preserves the coordinates $-n, \dots, -2$.

Thus $Z^{Zh}(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$ is an associative algebra with generators $e_{\pm 1, i}, e_{i, \pm 1}, i = -n, \dots, -2$, and also $e_{j, j}^{\pm 1}, j = -n, \dots, -2$. Here $e_{i, j} := p_{\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1}} E_{i, j}$.

This algebra has the following representations For an arbitrary representation \mathfrak{gl}_{n+1} the algebra $Z^{Zh}(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$ acts on the space of \mathfrak{gl}_{n-1} -highest vectors by the ruler

$$e_{i, j} \mapsto p_{\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1}} E_{i, j}, \quad e_{i, i}^{\pm 1} \mapsto E_{i, i}^{\pm 1},$$

Note that the action of $E_{i, i}^{-1}$ is well defined.

Define also elements $e_{-1, 1} = p_{\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1}} E_{-1, 1}, e_{1, -1} = p_{\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1}} E_{1, -1}$.

3.2.2 $Z^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$

Chose a subalgebra \mathfrak{sp}_{2n-2} that preserves the coordinates $\{-n, \dots, -2, 2, \dots, n\}$. The algebra $Z^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ is an associative algebra generated by $\zeta_{\pm 1, i}, \zeta_{i, \pm 1}, i = -n, \dots, -2$ and also $\zeta_{j, j}^{\pm 1}, j = -n, \dots, -2, 2, \dots, n$. Here $\zeta_{i, j} = p_{\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}} F_{i, j}$. This algebra has the following representations. For an arbitrary representation of \mathfrak{sp}_{2n} the algebra $Z^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ acts on the space of \mathfrak{sp}_{2n-2} -highest vectors by the ruler

$$\zeta_{i, j} \mapsto p_{\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}} F_{i, j}, \quad F_{i, i}^{\pm 1} \mapsto F_{i, i}^{\pm 1},$$

One can also add elements $\zeta_{i, j} = \text{sign}(i) \text{sign}(j) \zeta_{-j, -i}$.

Also put $\zeta_{-1, 1} = p_{\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}} F_{-1, 1}, \zeta_{1, -1} = p_{\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}} F_{1, -1}$.

3.2.3 $Z^M(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$

Let the algebra \mathfrak{gl}_{n+1} act in the space with coordinates $-n, -n+1, \dots, -1, n$, and let \mathfrak{gl}_{n-1} be a subalgebra that preserves the coordinates $-n+1, \dots, -1$.

Thus $Z^M(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$ is an associative algebra with generators $\varepsilon_{\pm n, i}, \varepsilon_{i, \pm n}, i = -n+1, \dots, -1$, and also $\varepsilon_{-n, n}, \varepsilon_{n, -n}, \varepsilon_{j, j}^{\pm 1}, j = -n+1, \dots, -1$. Here $\varepsilon_{i, j} = p_{\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1}} E_{i, j}$.

Also put $\varepsilon_{-n, n} = p_{\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1}} E_{-n, n}, \varepsilon_{n, -n} = p_{\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1}} E_{n, -n}$

3.2.4 $Z^M(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$

Chose a subalgebra \mathfrak{sp}_{2n-2} that preserves the coordinates $\{-n+1, \dots, n-1\}$. The algebra $Z(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ is an associative subalgebra generated by elements $\eta_{\pm n, i}, \eta_{i, \pm n}, i = -n+1, \dots, -1$, and also $\eta_{j, j}^{\pm 1}, j = -n+1, \dots, n-1$, where $\eta_{i, j} = p_{\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}} F_{i, j}$.

This algebra has the following representations. For an arbitrary representation of \mathfrak{sp}_{2n} the algebra $Z^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ acts on the space of \mathfrak{sp}_{2n-2} -highest vectors by the ruler

$$\eta_{i,j} \mapsto p_{\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}} F_{i,j}, \quad F_{i,i}^{\pm 1} \mapsto F_{i,i}^{\pm 1},$$

One can also add elements $\eta_{i,j} = \text{sign}(i)\text{sign}(j)\eta_{-j,-i}$.

Define also elements $\eta_{-n,n} = p_{\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}} F_{-n,n}$, $\eta_{n,-n} = p_{\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}} F_{n,-n}$.

For $i = 1, \dots, n$ define $\rho_{-i} = -\rho_i = i$, $f_i = F_{i,i} + \rho_i$ and $f_{-i} = -\rho_i$. Let $g_i = f_i + \frac{1}{2}$ for all i .

Define the operators

$$\check{\eta}_{i,\pm n} = \eta_{i,\pm n}(f_i - f_{i-1}) \dots (f_i - n + 1). \quad (7)$$

Define elements $Z_{n,-n}(u)$ by formula

$$Z_{n,-n}(u) = -F_{n,-n} \prod_{i=-n+1}^{n-1} (u + g_i) + \sum_{i=-n+1}^{n-1} \check{\eta}_{n,i} \check{\eta}_{i,n} \prod_{j=-n+1, j \neq i}^{n-1} \frac{u + g_j}{g_i - g_j} \quad (8)$$

3.3 The action of the Mikelsson-Zhelobenko algebra onto the determinants

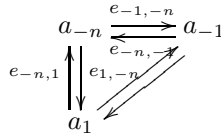
In the Corollary 1 below we show that in the restriction problems $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ and $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ in the Zhelobenko's approach the corresponding highest vector is a polynomial in $a_{-n}, \dots, -k$, $a_{-n}, \dots, -k-1, -1$, $a_{-n}, \dots, -k-1, 1$, $a_{-n}, \dots, -k-2, -1, 1$.

And in the Molev's approach - a polynomial in $a_{-n}, \dots, -k$, $a_{-n}, \dots, -k-1, n$, $a_{-n+1}, \dots, -k+1$, $a_{-n+1}, \dots, -k, n$.

Let us find some relations between operators, by means of which the generators of the Mikelsson-Zhelobenko algebra act on the determinants.

3.3.1 $Z^{Zh}(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$

The action of the generators onto determinants is described by the following diagram



An arrow that goes by diagonal up and right is $e_{1,-1}$, and an arrow that goes left and down is $e_{-1,1}$.

For $1 < k < n$ one has

$$\begin{array}{ccc}
 a_{-n}, \dots, -k & \xrightleftharpoons[e_{-k,-1}]{e_{-1,-k}} & a_{-n}, \dots, -k-1, -1 \\
 \updownarrow e_{-k,1} & \nearrow e_{1,-k} & \updownarrow -e_{1,-k-1} \\
 a_{-n}, \dots, -k-1, 1 & \xrightleftharpoons[e_{-k-1,-1}]{e_{1,-k-1}} & a_{-n}, \dots, -k-2, -1, 1
 \end{array}$$

An arrow that goes by diagonal up and right is $e_{1,-1}$, and an arrow that goes left and down is $e_{-1,1}$.

For $k = 1$

$$\begin{array}{ccc}
 & & a_{-n}, \dots, -2, -1 \\
 & \nearrow e_{-1,1} & \updownarrow -e_{1,-2} \\
 a_{-n}, \dots, -2, 1 & \xrightleftharpoons[e_{-2,-1}]{e_{1,-2}} & a_{-n}, \dots, -3, -1, 1
 \end{array}$$

3.3.2 $Z^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$

The action of the generators to the determinant is described by the following diagrams

$$\begin{array}{ccc}
 & & a_{-1} \\
 & \nearrow \zeta_{1,-n} & \\
 a_{-n} & \xrightleftharpoons[\zeta_{1,-n}]{\zeta_{-1,-n}} & a_{-1} \\
 \updownarrow \zeta_{-n,1} & & \\
 a_1 & &
 \end{array}$$

An arrow that goes by diagonal up and right is $\zeta_{1,-1}$, and an arrow that goes left and down is $\zeta_{-1,1}$.

For $1 < k < n$

$$\begin{array}{ccc}
 a_{-n}, \dots, -k & \xrightleftharpoons[\zeta_{-k,-1}]{\zeta_{-1,-k}} & a_{-n}, \dots, -k-1, -1 \\
 \updownarrow \zeta_{-k,1} & \nearrow \zeta_{1,-k} & \updownarrow -\zeta_{1,-k-1} \\
 a_{-n}, \dots, -k-1, 1 & \xrightleftharpoons[\zeta_{-k-1,-1}]{\zeta_{1,-k-1}} & a_{-n}, \dots, -k-2, -1, 1
 \end{array}$$

An arrow that goes by diagonal up and right is $\zeta_{1,-1}$, and an arrow that goes left and down is $\zeta_{-1,1}$.

For $k = 1$ we have the diagram

$$\begin{array}{ccc}
 & & a_{-n}, \dots, -2, -1 \\
 & \nearrow \zeta_{1,-2} & \updownarrow -\zeta_{1,-2} \\
 a_{-n}, \dots, -2, 1 & \xrightleftharpoons[\zeta_{-2,-1}]{\zeta_{1,-2}} & a_{-n}, \dots, -3, -1, 1
 \end{array}$$

An arrow that goes by diagonal up and right is $\zeta_{1,-1}$, and an arrow that goes left and down is $\zeta_{-1,1}$. A double arrow means that one determinant is mapped to another multiplied by 2.

To obtain this diagram we use the following fact. We have

$$\begin{aligned}\zeta_{1,-2}a_{-n,\dots,-2,-1} &= a_{-n,\dots,-3,1,-1} + a_{-n,\dots,-3,-2,2}, \\ \zeta_{-1,-2}a_{-n,\dots,-2,1} &= a_{-n,\dots,-3,-1,1} - a_{-n,\dots,-3,-2,2},\end{aligned}$$

But as it is shown below due to the relations between matrix elements in the group Sp_{2n} we have $a_{-n,\dots,-3,-2,2} = -a_{-n,\dots,-3,-1,1}$. This is the reason of the appearance of twin arrows.

Lemma 2. *For functions on Sp_{2n} one has a relation $a_{-n,\dots,-3,-2,2} = -a_{-n,\dots,-3,-1,1}$*

Proof. First of all we prove the equality for the group Sp_4 . It can be done as follows. Take a matrix $X \in Sp_4^0$ and consider its decomposition (1). The matrices ζ , δ and z can be represented as exponents of Lie algebra elements $\zeta = e^A$, $\delta = e^B$, $z = e^C$. The matrix A is a linear combination of $F_{i,j}$, $i > j$, B is a linear combination of $F_{i,i}$, C is a linear combination of $F_{i,j}$, $i < j$. Using this fact we obtain a parametrization of matrices A , B , C , then after taking an exponent a parametrization of matrices ζ , δ и z , and finally a presentation of an arbitrary matrix $X \in Sp_4^0$. Then we can check the equality $a_{-n,\dots,-3,-2,2} = -a_{-n,\dots,-3,-1,1}$ by direct computations³. Since Sp_4^0 is dense in Sp_4 the equality holds everywhere on Sp_4 .

Consider the case of an arbitrary group Sp_{2n} . Let $\mathcal{F}_{i,j}(\alpha)$ be a matrix with units on the diagonal, with α on the place (i,j) and with $-\alpha$ on the place $(-j,-i)$. The matrices $\mathcal{F}_{i,j}(\alpha)$ belong to Sp_{2n} . A multiplication by this matrix on the right is equivalent to doing an elementary transformation of the matrix X : we add to the j -th row the i -th row with a coefficient α , and we add simultaneously to the $-i$ -th row the $-j$ -th row with a coefficient $-\alpha$. A multiplication by $\mathcal{F}_{i,j}(\alpha)$ on the left is equivalent to an analogous transformation of rows.

Multiplying by $\mathcal{F}_{i,j}(\alpha)$ on the left and on the right we can transform X preserving $a_{-n,\dots,-3,-2,2}$ and $a_{-n,\dots,-3,-1,1}$ to the following form

³Of course it is better to do it using a computer

$$\begin{pmatrix}
x_{-n,-n} & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & x_{-n,n} \\
\dots & & & & & & & & & \\
0 & \dots & x_{-3,-3} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & x_{-2,-2} & x_{-2,-1} & x_{-2,1} & x_{-2,2} & 0 & \dots & 0 \\
0 & \dots & 0 & x_{-1,-2} & x_{-1,-1} & x_{-1,1} & x_{-1,2} & 0 & \dots & 0 \\
0 & \dots & 0 & x_{1,-2} & x_{1,-1} & x_{1,1} & x_{1,2} & 0 & \dots & 0 \\
0 & \dots & 0 & x_{2,-2} & x_{2,-1} & x_{2,1} & x_{2,2} & 0 & \dots & 0 \\
0 & \dots & x_{3,-3} & 0 & 0 & 0 & 0 & x_{3,3} & \dots & 0 \\
\dots & & & & & & & & & \\
x_{n,-n} & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & x_{n,n}
\end{pmatrix} \quad (9)$$

Since matrices $\mathcal{F}_{i,j}(\alpha)$ belong to Sp_{2n} , this matrix belongs to Sp_{2n} . We have

$$\begin{aligned}
a_{-n,\dots,-3,-2,2} &= x_{-n,-n} \dots x_{-3,-3} \cdot \det \begin{pmatrix} x_{-2,-2} & x_{-2,2} \\ x_{-1,-2} & x_{-1,2} \end{pmatrix}, \\
a_{-n,\dots,-3,-1,1} &= x_{-n,-n} \dots x_{-3,-3} \cdot \det \begin{pmatrix} x_{-2,-1} & x_{-2,1} \\ x_{-1,-1} & x_{-1,1} \end{pmatrix}.
\end{aligned}$$

The submatrix

$$\begin{pmatrix}
x_{-2,-2} & x_{-2,-1} & x_{-2,1} & x_{-2,2} \\
x_{-1,-2} & x_{-1,-1} & x_{-1,1} & x_{-1,2} \\
x_{1,-2} & x_{1,-1} & x_{1,1} & x_{1,2} \\
x_{2,-2} & x_{2,-1} & x_{2,1} & x_{2,2}
\end{pmatrix} \quad (10)$$

belongs to Sp_4 . Using the equality $a_{-2,2} = -a_{-1,1}$ for Sp_4 we obtain that $a_{-n,\dots,-3,-2,2}$ и $a_{-n,\dots,-3,-1,1}$ в Sp_{2n} .

□

3.3.3 $Z^M(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$

The action of the generators to the determinant is described by the following diagrams

$$\begin{array}{ccc}
& \xrightarrow{\varepsilon_{n,-n}} & a_n \\
a_{-n} & \xleftrightarrow{\varepsilon_{-n,n}} & \\
& \searrow \varepsilon_{-n+1,-n} & \\
\varepsilon_{-n,-n+1} \uparrow & & \\
& a_{-n+1} &
\end{array}$$

An arrow that goes by diagonal up and right is $\eta_{n,-n+1}$, and an arrow that goes left and down is $\eta_{-n+1,n}$.

For $1 < k < n$ we have

$$\begin{array}{ccc}
a_{-n}, \dots, -k & \xrightleftharpoons[\varepsilon_{-k,n}]{\varepsilon_{n,-k}} & a_{-n}, \dots, -k-1, n \\
\begin{array}{c} \varepsilon_{-n,-k+1} \updownarrow \varepsilon_{-k+1,-n} \end{array} & & \begin{array}{c} \varepsilon_{-n,-k} \updownarrow \varepsilon_{-k,-n} \end{array} \\
(-1)^{n-k-1} a_{-n+1}, \dots, -k, -k+1 & \xrightleftharpoons[\varepsilon_{-k+1,n}]{\varepsilon_{n,-k+1}} & (-1)^{n-k-1} a_{-n+1}, \dots, -k, n
\end{array}$$

An arrow that goes by diagonal up and right is $(-1)^{n-k-1} \varepsilon_{n,-n}$, and an arrow that goes left and down is $(-1)^{n-k-1} \varepsilon_{-n,n}$.

Now let $k = 1$.

$$\begin{array}{ccc}
a_{-n}, \dots, -1 & & \\
\begin{array}{c} \varepsilon_{-n,1} \updownarrow \varepsilon_{1,-n} \end{array} & & \\
(-1)^{n-1} a_{-n+1}, \dots, -1, 1 & \xrightleftharpoons[\varepsilon_{1,n}]{\varepsilon_{n,1}} & (-1)^{n-1} a_{-n+1}, \dots, -1, n
\end{array}$$

An arrow that goes by diagonal up and right is $(-1)^{n-1} \varepsilon_{n,-n}$, and an arrow that goes left and down is $(-1)^{n-1} \varepsilon_{-n,n}$.

3.3.4 $Z^M(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$

The action of the generators to the determinant is described by the following diagrams

$$\begin{array}{ccc}
a_{-n} & \xrightleftharpoons[\eta_{-n,n}]{\eta_{n,-n}} & a_n \\
\begin{array}{c} \eta_{-n,-n+1} \updownarrow \eta_{-n+1,-n} \end{array} & & \\
a_{-n+1} & &
\end{array}$$

An arrow that goes by diagonal up and right is $\eta_{n,-n+1}$, and an arrow that goes left and down is $\eta_{-n+1,n}$.

For $1 < k < n$ we have

$$\begin{array}{ccc}
a_{-n}, \dots, -k & \xrightleftharpoons[\eta_{-k,n}]{\eta_{n,-k}} & a_{-n}, \dots, -k-1, n \\
\begin{array}{c} \eta_{-n,-k+1} \updownarrow \eta_{-k+1,-n} \end{array} & & \begin{array}{c} \eta_{-n,-k} \updownarrow \eta_{-k,-n} \end{array} \\
(-1)^{n-k-1} a_{-n+1}, \dots, -k, -k+1 & \xrightleftharpoons[\eta_{-k+1,n}]{\eta_{n,-k+1}} & (-1)^{n-k-1} a_{-n+1}, \dots, -k, n
\end{array}$$

An arrow that goes by diagonal up and right is $(-1)^{n-k-1} \eta_{n,-n}$, and an arrow that goes left and down is $(-1)^{n-k-1} \eta_{-n,n}$.

Now let $k = 1$.

$$\begin{array}{ccc}
a_{-n, \dots, -1} & & \\
\eta_{-n,1} \updownarrow \eta_{1,-n} & \swarrow \searrow & \\
(-1)^{n-1} a_{-n+1, \dots, -1, 1} & \xrightleftharpoons[\eta_{1,n}]{\eta_{n,1}} & (-1)^{n-1} a_{-n+1, \dots, -1, n}
\end{array}$$

An arrow that goes by diagonal up and right is $(-1)^{n-1} \eta_{n,-n}$, and an arrow that goes left and down is $(-1)^{n-1} \eta_{-n,n}$.

3.3.5 The action of $Z_{n,-n}(u)$

Let us discuss the action of $Z_{n,-n}(u)$ onto determinants.

Lemma 3. *The operator $Z_{n,-n}(u)$ acts onto determinants by ruler*

$$\begin{aligned}
a_{-n, \dots, -i} &\mapsto \text{const} \cdot a_{-n+1, \dots, -i, n}, \\
\text{other determinants} &\mapsto 0,
\end{aligned}$$

and onto a product of determinants it acts by Leibnitz ruler. The constant depends on i and a formal variable u .

Proof. The first summand in (8) act as it is described in the statement of Lemma. Now consider the action of summands of type $\check{\eta}_{n,i} \check{\eta}_{i,-n}$ for $i = 1, \dots, n-1$. From the diagrams above one can make a conclusion that onto a determinant this composition acts by ruler

$$\begin{aligned}
a_{-n, \dots, -i} &\mapsto (-1)^{n-i} a_{-n+1, \dots, -i, n}, \\
\text{other determinants} &\mapsto 0,
\end{aligned}$$

and onto a product of determinants it acts by Leibnitz ruler.

□

3.4 A simplified Mikelsson-Zhelobenko algebra

Introduce three simplified Mikelsson-Zhelobenko algebras $Z_s^{Zh}(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$, $Z_s^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$, $Z_s^M(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$, $Z_s^M(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$.

The first two are algebras of operators that act on the space of polynomials in determinants $a_{-n, \dots, -k}$, $a_{-n, \dots, -k-1, -1}$, $a_{-n, \dots, -k-1, 1}$, $a_{-n, \dots, -k-2, -1, 1}$ and satisfy the Leibnitz ruler and commutation relations presented in the diagrams.

The third and the fourth algebras are constructed analogously but they are the algebras of operators that act on polynomials in variables $a_{-n, \dots, -k}$, $a_{-n, \dots, -k-1, n}$, $a_{-n+1, \dots, -k+1}$, $a_{-n+1, \dots, -k, n}$.

The simplified Mikelsson-Zhelobenko algebra is a factor of the ordinary Mikelsson-Zhelobenko algebra. The action of the ordinary Mikelsson-Zhelobenko algebra on the space of highest vectors factors through the homomorphism into the simplified Mikelsson-Zhelobenko algebra.

Let us compare the actions of $Z^{Zh}(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$ and $Z^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$, $Z^M(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$ and $Z^M(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$. We come to the following conclusion. Under the correspondence

$$\begin{aligned} e_{\pm 1, -2} &\leftrightarrow \frac{1}{2} \zeta_{\pm 1, -2} \\ e_{i,j} &\leftrightarrow \zeta_{i,j} \text{ in other cases} \end{aligned} \tag{11}$$

$$\begin{aligned} \varepsilon_{1, \pm n} &\leftrightarrow \frac{1}{2} \eta_{1, \pm n} \\ \varepsilon_{i,j} &\leftrightarrow \eta_{i,j} \text{ in other cases} \end{aligned} \tag{12}$$

the results of the actions on the determinants coincide.

Lemma 4. *The correspondences (11), (12) give isomorphisms of $Z_s^{Zh}(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$ and $Z_s^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$, $Z_s^M(\mathfrak{gl}_{n+1}, \mathfrak{gl}_{n-1})$ and $Z_s^M(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$. These isomorphism commute modulo multiplication onto a constant with the action on the space of \mathfrak{sp}_{2n-2} -highest vectors.*

Lemma 5. *The \mathfrak{sp}_{2n-2} -highest vector in the Zhelobenko's approach can be determined by formula (22), \mathfrak{sp}_{2n-2} -highest vector in the Molev's approach can be determined by formula (21)*

4 Explicit formulas for \mathfrak{sp}_{2n-2} -highest vectors

4.1 Zhelobenko's approach

Let us give explicit formulas for \mathfrak{sp}_{2n-2} -highest vectors in the Zhelobenko's approach.

$$v = \prod_{i=2}^n e_{-1, -i}^{m'_{-i, n} - m_{-i, n-1}} \prod_{i=1}^n e_{1, -i}^{m_{-i, n} - m'_{-i, n}} v_0, \tag{13}$$

where v_0 is the highest weight vector.

Using lemma 4 we come to the following conclusion

Lemma 6. *The \mathfrak{sp}_{2n-2} -c-highest vectors in the Zhelobenko's approach can be given by formula*

$$v = \prod_{i=2}^n \zeta_{-1,-i}^{m'_{-i,n} - m_{-i,n-1}} \prod_{i=1}^n \zeta_{1,-i}^{m_{-i,n} - m'_{-i,n}} v_0, \quad (14)$$

4.2 Molev's approach

Let us give formulas for \mathfrak{sp}_{2n-2} -highest vectors in the Molev's approach (see [6], [5]), corresponding to the tableau (3).

$$v = \prod_{i=1}^{n-1} \tilde{\eta}_{-n,-i}^{m'_{-i,n} - m_{-i,n-1}} \tilde{\eta}_{n,-i}^{m_{-i,n} - m'_{-i,n}} \prod_{k=m'_{-n,n} + \rho_n + \frac{1}{2}}^{m_{-n,n} + \rho_n + \frac{1}{2}} Z_{n,-n}(k) v_0 \quad (15)$$

Below we give a simplification of Molev's formulas.

In [6] there were obtained formulas for the action of $F_{n,-n}$ and $F_{-n,n}$. According to them after the action of the first operator onto a vector corresponding to a tableau we obtain a linear combination of vectors that are obtained by the change $m'_{-i,n} \mapsto m'_{-i,n} - 1$. And after the second - a linear combination of vectors that are obtained by the change $m'_{-i,n} \mapsto m'_{-i,n} + 1$.

Lemma 7. *The formula (15) can be simplified as follows.*

$$v = \text{const} \prod_{i=1}^{n-1} \eta_{-n,-i}^{m'_{-i,n} - m_{-i,n-1}} \prod_{i=1}^n \eta_{n,-i}^{m_{-i,n} - m'_{-i,n}} \quad (16)$$

Proof. To pass from (15) to (16) we first of all change $Z_{n,-n}(k)$ to $\eta_{n,-n}$. Indeed, from Lemma 3 it immediately follows that the results of the actions of $\eta_{n,-n}$ and $Z_{n,-n}(u)$ on the space \mathfrak{sp}_{2n-2} -highest vectors are proportional (the coefficient of proportionality depends on the vector)

Then using the commutation relations we change the order of factors and come to (16). □

Using Lemma 4 we obtain the following Lemma

Lemma 8. *\mathfrak{sp}_{2n-2} -highest vector in the Molev's approach can be given by the formula*

$$v = \text{const} \prod_{i=1}^{n-1} \varepsilon_{-n,-i}^{m'_{-i,n} - m_{-i,n-1}} \prod_{i=1}^n \varepsilon_{n,-i}^{m_{-i,n} - m'_{-i,n}} v_0 \quad (17)$$

From lemmas above we obtain the following result.

Corollary 1. *In the restriction problems $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ and $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ in the Zhelobenko's approach the corresponding highest vector is a polynomial in $a_{-n,\dots,-k}, a_{-n,\dots,-k-1,-1}, a_{-n,\dots,-k-1,1}, a_{-n,\dots,-k-2,-1,1}$.*

And in the Molev's approach - a polynomial in $a_{-n,\dots,-k}, a_{-n,\dots,-k-1,n}, a_{-n+1,\dots,-k+1}, a_{-n+1,\dots,-k,n}$.

5 Semimaximal diagrams

Definitio 1. We say that the tableau is *semimaximal in the Molev's approach* if it is of type

$$\begin{array}{cccccc} m_{-n,n} & m_{-n+1,n} & m_{-n+2,n} & \dots & m_{-1,n} & 0 \\ & m'_{-n,n} & m'_{-n+1,n} & m'_{-n+2,n} & \dots & m'_{-1,n} \\ & & m'_{-n+1,n} & m'_{-n+2,n} & \dots & m'_{-1,n}, \end{array} \quad (18)$$

that is if $m'_{-i+1,n} = m_{-i,n-1}$.

Definitio 2. We say that the tableau is *semimaximal in the Zhelobenko's approach* if it is of type

$$\begin{array}{cccccc} m_{-n,n} & m_{-n+1,n} & m_{-n+2,n} & \dots & m_{-1,n} & 0 \\ & m'_{-n,n} & m'_{-n+1,n} & m'_{-n+2,n} & \dots & m'_{-1,n} \\ & m'_{-n,n} & m'_{-n+1,n} & m'_{-n+2,n} & \dots & \end{array} \quad (19)$$

that is if $m'_{-i,n} = m_{-i,n-1}$.

Let us be give a tableau of type (3). Then the corresponding \mathfrak{sp}_{2n-2} -highest vector in the Molev's approach is of type

$$(m)^M := \text{const} \prod_{i=1}^{n-1} \eta_{n,i}^{m_{-i,n-1}-m'_{-i+1,n}} v_1, \quad (20)$$

where v_1 - is a vector of the semimaximal tableau that is obtained by the change of the lower row $m_{-i,n-1} \mapsto m'_{-i+1,n}$.

From Lemma 4 it follows that

$$(m)^M := \text{const} \prod_{i=1}^{n-1} \varepsilon_{n,i}^{m_{-i,n-1}-m'_{-i+1,n}} v_1, \quad (21)$$

Indeed as it is shown in the next section the action of $\varepsilon_{n,i}$ and $\eta_{n,i}$ differs only by a constant.

An analogous formula is true in the Zhelobenko's approach, the \mathfrak{sp}_{2n-2} -highest vector can be represented as follows

$$(m)^{Zh} := \text{const} \prod_{i=1}^{n-1} e_{-1,-i}^{m_{-i,n-1}-m'_{-i+1,n}} v_2, \quad (22)$$

where v_2 - is a vector of the semimaximal tableau that is obtained by the change of the lower row $m_{-i,n-1} \mapsto m'_{-i,n}$.

From Lemma 4 it follows that

$$(m)^{Zh} := \text{const} \prod_{i=1}^{n-1} \zeta_{-1,-i}^{m'_{-i,n}-m_{-i,n-1}} v_2, \quad (23)$$

6 The main theorem

Definitio 3. Let us be given a tableau (m) of type (3), define the tableau (\bar{m}) by formula

$$\begin{array}{ccccccc} m_{-n,n} & m_{-n+1,n} & \dots & m_{-1,n} & 0 & & \\ m'_{-n,n} & m'_{-n+1,n} & \dots & m'_{-1,n} & & & \\ \bar{m}_{-n,n-1} & \dots & \bar{m}_{-2,n-1} & & & & \end{array} \quad (24)$$

Where $\bar{m}_{-i,n-1} = m'_{-i,n} - (m_{-i,n-1} - m'_{-i+1,n})$.

Note that if a tableau (m) is semimaximal in the Zhelobenko's sense that (\bar{m}) is semimaximal in the Molev's sense. Also $(\bar{\bar{m}}) = (m)$

Theorem 1. Under the correspondence $(m)^{Zh} \leftrightarrow (\bar{m})^M$ the actions of $Z_s^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ and $Z_s^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ are conjugated modulo a constant by the ruler

$$\begin{aligned} \zeta_{1,-k} &\sim \eta_{n, -(n-k+1)}, & \zeta_{-k,1} &\sim \eta_{-(n-k+1),n} \\ \zeta_{-1,-k} &\sim \eta_{n-k+1,n}, & \zeta_{-k,-1} &\sim \eta_{n,n-k+1} \end{aligned}$$

6.1 The proof in the case of semimaximal tableaux

Let us prove a statement analogous to the main statement of Section 2.2.

Lemma 9. Let us be given the algebra \mathfrak{gl}_{n+1} acting in the space with coordinates $-n, -n+1, \dots, -1, n$, and let us be given a subalgebra \mathfrak{gl}_{n-1} that preserves coordinates $-n+1, \dots, -1$. Also let \mathfrak{sp}_{2n} act in the space with coordinates $-n, -n+1, \dots, n-1, n$, and let \mathfrak{sp}_{2n-2} preserve the coordinates $-n+1, \dots, n-1$.

Then the problems of restriction $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ and $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ are equivalent.

Proof. Indeed take an embedding of \mathfrak{gl}_{n+1} into \mathfrak{gl}_{2n} as a subalgebra that preserve coordinates $-n, -n+1, \dots, -1, n$. Then \mathfrak{gl}_{n+1} -highest vectors are presented as polynomials in variables $z_{-n, -n+1}, \dots, z_{-n, -1}$, and $z_{-n, n}, z_{-n+1, n}, \dots, z_{-1, n}$.

But in the problem of restriction $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ the \mathfrak{sp}_{2n-2} -highest vectors are presented as polynomials in the same variables.

Also the indicator systems for these polynomials in the problems of restriction $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ and $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ coincide.

□

Using this Lemma and Lemma 8 we come to an important conclusion that the Molev's base can be obtained by a procedure analogous to the procedure that defines the Zhelobenko's base but using another restriction $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$.

Corollary 2. *Under the equivalence of restriction problems established in Corollary 3 the Gelfand-Tsetlin base for $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ is mapped modulo a constant to the Molev's base in the restriction problem $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$.*

Lemma 10. *In the realization of the representation in the function on the group Sp_{2n} in the Zhelobenko's approach a base vector corresponding to (19) is a polynomial*

$$\text{const} \prod_{k=1}^{n-1} a_{-n, \dots, -k+1, 1}^{m_{-(n-k), n} - m'_{-(n-k), n}} a_{-n, \dots, -k}^{m'_{-(n-k), n} - m_{-(n-k)+1, n}},$$

and in the Molev's approach the vector corresponding to (18) is the polynomial

$$\text{const} \prod_{k=1}^{n-1} a_{-n, \dots, -k+1, n}^{m_{-k, n} - m'_{-k, n}} a_{-n, \dots, -k}^{m'_{-k, n} - m_{-k, n}}.$$

Proof. The statement of the Lemma in the case of Zhelobenko's approach follows from the formula of \mathfrak{gl}_n -highest vectors in a \mathfrak{gl}_{n+1} -representation (see [4]).

Let us prove the Lemma in the case of Molev's approach. Using Lemma 4 we obtain that the vector corresponding to a semimaximal diagram is given by formula

$$\text{const} \prod_{i=1}^{n-1} \varepsilon_{n, -i}^{m_{i, n} - m'_{i, n}} v_0.$$

Using the formulas for the action of $\varepsilon_{n, i}$ onto the determinants we immediately obtain the statement of the Lemma.

□

Corollary 3. *In the realization of the representation in the function on the group Z in the Zhelobenko's approach a base vector corresponding to (19) is a*

polynomial

$$\text{const} \prod_{k=1}^{n-1} z_{-k,1}^{m_{-(n-k),n} - m'_{-(n-k),n}},$$

and in the Molev's approach the vector corresponding to (18) is the polynomial

$$\text{const} \prod_{k=1}^{n-1} z_{-(n-k+1),n}^{m_{-k,n} - m'_{-k,n}}.$$

Now let us return to the proof of the Theorem in the case of semi-maximal tableaux. For semimaximal tableaux the correspondence $(m)^{Zh} \leftrightarrow (\bar{m})^M$ means the correspondence

$$\begin{aligned} a_{-n,\dots,-k} &\leftrightarrow a_{-n,\dots,-k} \\ a_{-n,\dots,-k+1,1} &\leftrightarrow a_{-n,\dots,-n+k,n} \end{aligned} \quad (25)$$

Thus we need to prove the following statement.

Lemma 11. *The actions of $\zeta_{1,-k}$ and $\eta_{n,-(n-k+1)}$, $\zeta_{-k,1}$ and $\eta_{-(n-k+1),n}$, $k = n, \dots, 1$ on the semimaximal tableaux under the correspondence (25) are conjugated modulo multiplication by a constant.*

Proof. It is sufficient to prove that the actions of $\zeta_{1,-k}$ and $\eta_{n,-(n-k)}$, $\zeta_{-k,1}$ and $\eta_{-(n-k+1),n}$ on $a_{-n,\dots,-p}$, $a_{-n,\dots,-p-1,1}$ and $a_{-n,\dots,-p}$, $a_{-n,\dots,-p-1,n}$ are conjugated under the correspondence (25)

Under the action of $\zeta_{1,-k}$ the determinant are changed as follows

$$\begin{aligned} a_{-n,\dots,-p} &\mapsto \begin{cases} 0, & p \neq k \\ a_{-n,\dots,-p-1,1}, & p = k \end{cases} \\ a_{-n,\dots,-p-1,1} &\mapsto 0 \end{aligned}$$

Under the action of $\zeta_{-k,1}$ the determinant are changed as follows

$$\begin{aligned} a_{-n,\dots,-p} &\mapsto 0 \\ a_{-n,\dots,-p-1,1} &\mapsto \begin{cases} 0, & p \neq k \\ a_{-n,\dots,-p-1,-p}, & p = k \end{cases} \end{aligned}$$

Under the action of $\eta_{n,-(n-k+1)}$ the determinant are changed as follows

$$a_{-n,\dots,-p} \mapsto \begin{cases} 0, & p \neq n-k+1 \\ a_{-n,\dots,-p-1,n}, & p = n-k+1 \end{cases}$$

$$a_{-n,\dots,-p-1,n} \mapsto 0$$

Under the action of $\eta_{-(n-k+1),n}$ the determinants are changed as follows

$$\begin{aligned} a_{-n,\dots,-p} &\mapsto 0 \\ a_{-n,\dots,-p-1,n} &\mapsto \begin{cases} 0, & p \neq n-k+1 \\ a_{-n,\dots,-p-1,-p}, & p = n-k+1 \end{cases} \end{aligned}$$

If one compares the corresponding formulas, then using Lemma 10 one obtains the conclusion of the Lemma. \square

6.2 The proof in the case of an arbitrary tableau

We use the formulas (20) and (23).

It is sufficient to check that the following actions are conjugated

$$\begin{aligned} \eta_{n-k+1,n}(\bar{m})^M &\leftrightarrow \zeta_{-1,-k}(m)^{Zh}, & \eta_{n,n-k+1}(\bar{m})^M &\leftrightarrow \zeta_{-k,-1}(m)^{Zh} \\ \eta_{n,-(n-k+1)}(\bar{m})^M &\leftrightarrow \zeta_{-k,n}(m)^{Zh}, & \eta_{-(n-k+1),n}(\bar{m})^M &\leftrightarrow \zeta_{n,-k}(m)^{Zh}. \end{aligned}$$

6.2.1 The proof for $\zeta_{-1,-k}$ and $\eta_{n-k+1,n}$

Since the operators $\zeta_{-1,-i}$ commute, the multiplication of (23) onto $\zeta_{-1,-k}$ means that we make the following change

$$m'_{-k,n} - m_{-k,n-1} \mapsto m'_{-k,n} - m_{-k,n-1} + 1.$$

Analogously the multiplication of (20) onto $\eta_{n-k+1,n}$ is equivalent to the change

$$m'_{-k,n} - m_{-k,n-1} \mapsto m'_{-k,n} - m_{-k,n-1} + 1.$$

Thus both operations make a change

$$m_{-k,n-1} \mapsto m_{-k,n-1} - 1,$$

Hence these operations are conjugate

6.2.2 The proof for $\zeta_{-k,-1}$ and $\eta_{n,n-k+1}$

In this case we must compare the results of multiplication of (23) on $\zeta_{-k,-1}$ and the result of multiplication of (20) on $\eta_{k,n}$.

The operators $\zeta_{-1,-i}$, $i > k$ and $\zeta_{-k,-1}$ commute and $\zeta_{-k,-1}\zeta_{-1,-k} = 1$. Thus the multiplication of (23) onto $\zeta_{-k,-1}$ means that we make a change

$$m'_{-k,n} - m_{-k,n-1} \mapsto m'_{-k,n} - m_{-k,n-1} - 1.$$

analogously the multiplication of (20) onto $\eta_{n,n-k+1}$ is equivalent to the change

$$m'_{-k,n} - m_{-k,n-1} \mapsto m'_{-k,n} - m_{-k,n-1} - 1.$$

Thus both operations make a change

$$m_{-k,n-1} \mapsto m_{-k,n-1} + 1,$$

Hence these operations are conjugate.

6.2.3 The prof for $\zeta_{1,-k}$ and $\eta_{-(n-k+1),n}$.

We must compare the results of multiplication of (23) onto $\zeta_{1,-k}$ and the result of multiplication (20) onto $\eta_{-(n-k+1),n}$.

Consider the first expression. Considering the defining commutation relations for the algebra $Z_s^{zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ one obtains a relation

$$\zeta_{1,-k}\zeta_{-1,-k+1} = \zeta_{-1,-k}\zeta_{1,-k+1}.$$

Let in the formula (23) the factor $\zeta_{-1,-k+1}$ enters with a nonzero exponent $b = m'_{-(k+1),n} - m_{-(k+1),n-1} > 0$, then

$$\begin{aligned} \zeta_{1,-k}(m)^{Zh} &= \zeta_{1,-k}(\dots \zeta_{-1,-k}^a \zeta_{-1,-k+1}^b \dots) v_1 = (\dots \zeta_{-1,-k}^a \zeta_{1,-k} \zeta_{1,-k+1}^b \dots) v_1 = \\ &= (\dots \zeta_{-1,-k}^a \zeta_{1,-k} \zeta_{-1,-k} \zeta_{-1,-k}^{b-1} \dots) v_1 = (\dots \zeta_{-1,-k}^a \zeta_{-1,-k} \zeta_{1,-k+1} \zeta_{-1,-k+1}^{b-1} \dots) v_1 = \\ &= (\dots \zeta_{-1,-k}^{a+1} \zeta_{-1,-k+1}^{b-1} \dots) \zeta_{-1,-k+1} v_1 \end{aligned} \tag{26}$$

And if $b = 0$, then

$$\zeta_{1,-k}(m)^{Zh} = \zeta_{1,-k}(\dots) v_1 = (\dots) \zeta_{1,-k} v_1 \tag{27}$$

Also one has in $Z_s^M(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ a relation

$$\eta_{-k,n} \eta_{n,k+1} = \eta_{n,k} \eta_{-k-1,n}.$$

As a corollary we get a relation

$$\eta_{-(n-k+1),n} \eta_{n,(n-k+2)} = \eta_{n,(n-k+1)} \eta_{-(n-k+2),n}.$$

Let in the formula (20) the factor $\eta_{n,k}$ enters with a nonzero exponent $b = m'_{-(k-1),n} - m_{-(k-1),n-1} > 0$, then

$$\begin{aligned}
\eta_{-(n-k+1),n}(\bar{m})^M &= \eta_{-(n-k+1),n}(\dots\eta_{n,(n-k+1)}^a\eta_{n,(n-k+2)}^b\dots)v_2 = \\
&= (\dots\eta_{n,(n-k+1)}^a\eta_{-(n-k+1),n}\eta_{n,(n-k+2)}^b\dots)v_2 = \\
&= (\dots\eta_{n,(n-k+1)}^a\eta_{-(n-k+1),n}\eta_{n,(n-k+2)}\eta_{n,(n-k+2)}^{b-1}\dots)v_2 = \\
&= (\dots\eta_{n,(n-k+1)}^a\eta_{n,(n-k+1)}\eta_{-(n-k+2),n}\eta_{n,(n-k+2)}^{b-1}\dots)v_2 = \\
&= (\dots\eta_{n,(n-k+1)}^{a+1}\eta_{n,(n-k+2)}^{b-1}\dots)\eta_{-(n-k+2),n}v_2
\end{aligned} \tag{28}$$

And if b , then

$$\eta_{-(n-k+1),n}(\bar{m})^M = \eta_{-(n-k+1),n}(\dots)v_2 = (\dots)\eta_{-(n-k+1),n}v_2 \tag{29}$$

Compare the formulas (26) and (28), (27) and (29). We come to the following result. If $m'_{-(k-1),n} - m_{-(k-1),n-1} \neq 0$, then in both cases the following changes occur. Firstly $m_{-(k-1),n-1} \mapsto m_{-(k-1),n-1} + 1$, secondly onto v_1 and v_2 there acts $\zeta_{1,-k+1}$ or $\eta_{-(n-k+2),n}$ correspondingly. And if $m'_{-(k-1),n} - m_{-(k-1),n-1} = 0$, then onto v_1 and v_2 there acts $\zeta_{1,-k}$ or $\eta_{-(n-k+1),n}$ correspondingly.

Hence both actions are conjugated modulo multiplication by a constant under the considered correspondence.

6.2.4 The proof for $\zeta_{-k,1}$ and $\eta_{n,-(n-k+1)}$.

We must compare the results of multiplication of (23) onto $\zeta_{-k,1}$ and the result of multiplication of (20) onto $\eta_{n,-k}$.

Consider the first expression, one in has the simplified algebra of Mikelsson-Zhelobenko $Z_s^{Zh}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ a relation

$$\zeta_{-k,1}\zeta_{-1,-k} = \zeta_{-1,-(k-1)}\zeta_{-(k-1),1}.$$

Let in the formula (23) the factor $\zeta_{1,k+1}$ occurs with a nonzero exponent $b = m'_{-k,n} - m_{-k,n-1} > 0$ then

$$\begin{aligned}
\zeta_{-k,1}(m)^{Zh} &= \zeta_{-k,1}(\dots\zeta_{-1,-k}^b\zeta_{-1,-(k-1)}^a\dots)v_1 = (\dots\zeta_{-k,1}\zeta_{-1,-k}^b\zeta_{-1,-(k-1)}^a\dots)v_1 = \\
&= (\dots\zeta_{-k,1}\zeta_{-1,-k}\zeta_{-1,-k}^{b-1}\zeta_{-1,-(k-1)}^a\dots)v_1 = (\dots\zeta_{-1,-(k-1)}\zeta_{-(k-1),1}\zeta_{-1,-k}^{b-1}\zeta_{-1,-(k-1)}^a\dots)v_1 = \\
&= (\dots\zeta_{-1,-k}^{b-1}\zeta_{-1,-(k-1)}^{a+1}\dots)\zeta_{-(k-1),1}v_1
\end{aligned} \tag{30}$$

And if $b = m'_{-k,n} - m_{-k,n-1} = 0$ then

$$\zeta_{-k,1}(m)^{Zh} = \zeta_{-k,1}(\dots)v_1 = (\dots)\zeta_{-k,1}v_1 \tag{31}$$

From the other hand one has in $Z_s^M(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ a relation

$$\eta_{n,-k}\eta_{n,k} = \eta_{n,k+1}\eta_{n,-(k+1)}.$$

As a corollary we obtain a relation

$$\eta_{n,-(n-k+1)}, \eta_{n,(n-k+1)} = \eta_{n,(n-k+2)}\eta_{n,-(n-k+2)}.$$

Let in the formulas (20) the factor $\eta_{n,(n-k+1)}$ occurs with a nonzero exponent $b = m'_{-k,n} - m_{-k,n-1} > 0$ then

$$\begin{aligned} \eta_{n,-(n-k+1)}(\bar{m})^M &= \eta_{n,-(n-k+1)}(\dots\eta_{n,(n-k+1)}^b\eta_{n,(n-k+2)}^a\dots)v_2 = \\ &= (\dots\eta_{n,-(n-k+1)}\eta_{n,(n-k+1)}^b\eta_{n,(n-k+2)}^a\dots)v_2 = \\ &= (\dots\eta_{n,-(n-k+1)}\eta_{n,(n-k+1)}^{b-1}\eta_{n,(n-k+2)}^a\dots)v_2 = \\ &= (\dots\eta_{n,(n-k+2)}\eta_{n,-(n-k+2)}\eta_{n,(n-k+1)}^{b-1}\eta_{n,(n-k+2)}^a\dots)v_2 = \\ &= (\dots\eta_{n,(n-k+1)}^{b-1}\eta_{n,(n-k+2)}^{a+1}\dots)\eta_{n,-k+1}\eta_{n,(n-k+2)}v_2 \end{aligned} \quad (32)$$

If $b = m'_{-k,n} - m_{-k,n-1} = 0$ then

$$\eta_{n,-(n-k+1)}(\bar{m})^M = \eta_{n,-(n-k+1)}(\dots)v_2 = (\dots)\eta_{n,-(n-k+1)}v_2 \quad (33)$$

Compare the formulas (30) and (32), (31) and (33). We come to the following result. If $m'_{-k,n} - m_{-k,n-1} \neq 0$, then in both cases the following changes occur. Firstly $m_{-k,n-1} \mapsto m_{-k,n-1} + 1$, secondly onto v_1 and v_2 there acts $\zeta_{1,-(k-1)}$ or $\eta_{n,-(n-k+2)}$ correspondingly. And if $m'_{-k,n} - m_{-k,n-1} = 0$, then onto v_1 and v_2 there acts $\zeta_{1,-k}$ or $\eta_{n,-(n-k+1)}$ correspondingly.

Hence both actions are conjugated modulo multiplication by a constant under the considered correspondence.

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